

# THE CLOSURE OF THE SYMPLECTIC CONE OF ELLIPTIC SURFACES

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**ABSTRACT.** The symplectic cone of a closed oriented 4-manifold is the set of cohomology classes represented by symplectic forms. A well-known conjecture describes this cone for every minimal Kähler surface. We consider the case of the elliptic surfaces  $E(n)$  and focus on a slightly weaker conjecture for the closure of the symplectic cone. We prove this conjecture in the case of the spin surfaces  $E(2m)$  using inflation and the action of self-diffeomorphisms of the elliptic surface. An additional obstruction appears in the non-spin case.

## 1. INTRODUCTION

Let  $M$  be a closed oriented 4-manifold. We are interested in the set  $\mathcal{C}_M$  of real cohomology classes represented by symplectic forms on  $M$ , called the *symplectic cone* of  $M$ . It is indeed a cone because a non-zero multiple of any symplectic form is again symplectic. We only consider symplectic forms  $\omega$  compatible with the orientation, so that  $\omega \wedge \omega$  is everywhere positive. It follows that the symplectic cone is a subset of the *positive cone*  $\mathcal{P}$ , given by the set of elements in  $H^2(M; \mathbb{R})$  which have positive square. If the 4-manifold  $M$  does not admit a symplectic form then the set  $\mathcal{C}_M$  is empty. It is also useful to denote by  $\mathcal{P}^A$  for a non-zero cohomology class  $A \in H^2(M; \mathbb{R})$  the set of elements in  $\mathcal{P}$  which have positive cup product with  $A$ . In addition, we set  $\mathcal{P}^0 = \mathcal{P}$ .

The symplectic cone has been determined in the following cases:

- (a)  $S^2$ -bundles over surfaces [17].
- (b)  $T^2$ -bundles over  $T^2$  [8].
- (c) All 4-manifolds with a fixed point free circle action [3, 5, 6].
- (d) All symplectic 4-manifolds with  $b_2^+ = 1$  [14].
- (e) The  $K3$  surface [13].
- (f) Fibre sums along tori of  $T^2 \times \Sigma_g$  and minimal elliptic Kähler surfaces with  $b_2^+ = 1$ , for example Enriques or Dolgachev surfaces [4].

The simply-connected 4-manifolds among these cases either have  $b_2^+ = 1$  or are diffeomorphic to the  $K3$  surface, because the 4-manifolds in (c) have zero Euler characteristic.

From now on we denote by  $M$  a simply-connected elliptic surface  $E(n)$  without multiple fibres and with the complex orientation. Since by the results mentioned

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above the symplectic cone is known for  $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$  and the  $K3$  surface  $E(2)$  we assume that  $n \geq 3$ . Let  $F$  denote the class of the fibre in an elliptic fibration on  $M$ . We set

$$c_1(M) = -(n-2)PD(F),$$

where  $PD$  denotes the Poincaré dual of the homology class. Note that symplectic forms have well-defined Chern classes, defined by considering any compatible almost complex structure. If  $\omega$  is a symplectic form with first Chern class  $c_1(M, \omega)$ , then  $-\omega$  is a symplectic form with first Chern class  $-c_1(M, \omega)$ . It is known from Seiberg-Witten theory [10] that every symplectic form on  $E(n)$  has up to sign first Chern class equal to  $c_1(M)$ . It is also known from the theorems of Taubes [18] that for every symplectic structure  $\omega$  the class  $-c_1(M, \omega)$  is represented by an embedded symplectic surface. This implies that  $\omega \cdot c_1(M)$  is non-zero, hence the symplectic cone satisfies

$$\mathcal{C}_M \subset \left( \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)} \right).$$

A well-known conjecture due to Tian-Jun Li [13] says that the following holds:

**Conjecture 1** (Strong conjecture). *We have*

$$\left( \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)} \right) \subset \mathcal{C}_M.$$

*Hence every class of positive square whose cup product with the first Chern class of  $M$  is non-zero is represented by a symplectic form.*

The conjecture should even hold for any closed 4-manifold underlying a minimal Kähler surface, but we only consider the case of elliptic surfaces. There is also a slightly weaker form of the conjecture. We denote by  $\overline{\mathcal{C}}_M$  the closure of the symplectic cone in the vector space  $H^2(M; \mathbb{R})$ .

**Conjecture 2** (Weak conjecture). *We have*

$$\left( \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)} \right) \subset \overline{\mathcal{C}}_M.$$

*Hence every class of positive square whose cup product with the first Chern class of  $M$  is non-zero is the limit of a sequence of symplectic classes.*

In the following we only consider the weak conjecture. To state the theorem we want to prove, consider the following definition:

**Definition 1.** We define

$$\mathcal{P}^{c_1(M)>} \subset \mathcal{P}^{c_1(M)}$$

and

$$\mathcal{P}^{-c_1(M)>} \subset \mathcal{P}^{-c_1(M)}$$

to be the subsets of elements  $\omega$  with  $\omega^2 > (\omega \cdot PD(F))^2$ .

Then we have:

**Theorem 2.** *Let  $m \geq 2$  be an integer. If  $M$  is the spin surface  $E(2m)$  then*

$$\left( \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)} \right) \subset \overline{\mathcal{C}}_M.$$

*If  $M$  is the non-spin surface  $E(2m - 1)$  then*

$$\left( \mathcal{P}^{c_1(M)>} \cup \mathcal{P}^{-c_1(M)>} \right) \subset \overline{\mathcal{C}}_M.$$

This proves Conjecture 2 in the case of the spin elliptic surfaces  $E(2m)$ . At the moment we do not know how to prove the full Conjecture 2 in the non-spin case. One can view these results as evidence that the strong Conjecture 1 is indeed true. The sequences of symplectic forms in the theorem are all obtained from a single symplectic form by inflation along certain symplectic surfaces and the action of the orientation preserving self-diffeomorphisms of the elliptic surface  $M$ .

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## 2. SOME NOTATION

We follow the notation from [11]. In particular, all self-diffeomorphisms of  $M$  are orientation preserving. We often denote a symplectic form and its class by the same symbol. Note that considering minus a given symplectic form we see that to prove the weak conjecture it suffices to prove that

$$\mathcal{P}^{PD(F)} \subset \overline{\mathcal{C}}_M.$$

**Definition 3.** We define

$$\mathcal{P}^{PD(F)>} \subset \mathcal{P}^{PD(F)}$$

to be the subset of elements  $\omega$  with  $\omega^2 > (\omega \cdot PD(F))^2$ .

We want to prove the following theorem, which is equivalent to Theorem 2:

**Theorem 4.** *Let  $m \geq 2$  be an integer. If  $M$  is the spin surface  $E(2m)$  then*

$$\mathcal{P}^{PD(F)} \subset \overline{\mathcal{C}}_M.$$

*If  $M$  is the non-spin surface  $E(2m - 1)$  then*

$$\mathcal{P}^{PD(F)>} \subset \overline{\mathcal{C}}_M.$$

We will first prove a special case of Theorem 4 since this is easier and uses the same method as in the general case. We need some notation. Consider the manifold  $M = E(n)$  and define an integer  $m$  by  $n = 2m$  if  $n$  is even and  $n = 2m - 1$  if  $n$  is odd.

**Definition 5.** Let  $W$  be the embedded surface obtained by smoothing the intersections of a section  $V$  of the elliptic surface  $M$  of square  $-n$  and  $m$  parallel copies of the fibre  $F$ . Let  $R$  denote a rim torus of square zero and  $S$  a dual vanishing sphere of square  $-2$  in  $M$ . Both  $F, W$  and  $R, S$  intersect in a single transverse positive point. Otherwise the surfaces are disjoint.

The vanishing sphere  $S$  is obtained by sewing together in the fibre sum

$$E(n) = E(1) \#_{F=F} E(n-1)$$

two vanishing disks coming from singular fibres with the same vanishing cycles. The surface  $W$  has self-intersection zero if  $n$  is even and one if  $n$  is odd. The surfaces  $F$  and  $W$  span a copy of the standard hyperbolic form  $H$  in the intersection form if  $n$  is even and a copy of  $H'$  if  $n$  is odd, where

$$H' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We will denote both intersection forms by  $H(n)$ . Since this form is unimodular the total intersection form over the integers looks like

$$Q_M = H(n) \oplus H(n)^\perp.$$

We can decompose any class  $\omega \in H^2(M; \mathbb{R})$  according to this splitting as

$$\omega = PD(\alpha F + \beta W) + \omega'$$

with  $\omega' \in H(n)^\perp$  (here we mean the real subspace spanned by this lattice). Note that

$$\beta = \omega \cdot PD(F).$$

**Definition 6.** We define

$$\mathcal{P}^{PD(F)+} \subset \mathcal{P}^{PD(F)}$$

to be the subset of elements of the form  $\omega = PD(\alpha F + \beta W) + \omega'$  where  $\alpha, \beta$  and  $\omega'^2$  are positive. We call the classes in this subset *positive*.

**Theorem 7.** *We have*

$$\mathcal{P}^{PD(F)+} \subset \overline{\mathcal{C}}_M.$$

This theorem describes the first subset of  $\mathcal{P}^{PD(F)}$  that we can represent by the limits of symplectic forms. We will extend it later and prove Theorem 4.

### 3. SYMPLECTIC FORMS AND DIFFEOMORPHISMS

The inflation procedure, introduced by Lalonde and McDuff [12, 17], shows that if  $\Sigma$  is a closed connected symplectic surface of non-negative square in a closed symplectic 4-manifold  $(Y, \omega)$ , then the class  $[\omega] + tPD(\Sigma)$  is represented by a symplectic form for all  $t \geq 0$ . We need the following generalized inflation lemma:

**Lemma 8.** *Let  $(Y, \omega)$  be a closed symplectic 4-manifold and  $\Sigma_1, \Sigma_2 \subset Y$  closed connected symplectic surfaces of non-negative square which intersect transversely in a single positive point. Then for all real numbers  $r_1, r_2 \geq 0$  the class*

$$[\omega] + r_1 PD(\Sigma_1) + r_2 PD(\Sigma_2)$$

*is represented by a symplectic form.*

*Proof.* By the symplectic neighbourhood theorem  $\Sigma_1$  has a tubular neighbourhood  $\nu\Sigma_1$  with symplectic fibres. According to Lemma 2.3 in [9] we can assume that  $\Sigma_2$  intersects  $\nu\Sigma_1$  in one of the disk fibres. If we first do inflation along  $\Sigma_1$  as in [17, Lemma 3.7] then the symplectic form changes only in the tubular neighbourhood  $\nu\Sigma_1$  and the fibres stay symplectic. Hence  $\Sigma_2$  remains symplectic and we can then do inflation along  $\Sigma_2$ . Compare with [2, Lemma 2.1.A] and [15, Theorem 2.3].  $\square$

**Proposition 9.** *There exists a symplectic form on  $M$  such that  $F, W, R$  and  $S$  are symplectic surfaces.*

From the Gompf sum construction [9] applied to the fibre sum

$$E(n) = E(1) \#_{F=F} E(n-1)$$

it is clear that there exists a symplectic form on  $M$  such that  $F$  and  $V$  are symplectic. Hence the surface  $W$  is also symplectic.

**Lemma 10.** *We can choose the surfaces  $R$  and  $S$  such that they are Lagrangian for a symplectic form from the Gompf construction.*

*Proof.* The claim is clear for the rim torus  $R$ : In the fibre sum construction it is given by  $R = \gamma \times \partial D^2$  where  $\gamma$  is one of the circle factors of the torus  $F = S^1 \times S^1$  in a tubular neighbourhood  $F \times D^2$  on which the symplectic form is a standard product form. The claim for the vanishing sphere  $S$  follows from section 8 in [1].  $\square$

Hence Proposition 9 is a consequence of the following theorem that we formulate in a more general way. The proof is very similar to Lemma 1.6 in [9] due to Gompf which states the same for disjoint Lagrangians.

**Theorem 11.** *Let  $(X, \omega)$  be a closed symplectic 4-manifold and  $L_1, \dots, L_n$  closed connected embedded oriented Lagrangian surfaces in  $X$  which intersect each other transversely so that at most two surfaces intersect in any given point of  $X$ . Suppose that the classes of these surfaces are linearly independent in  $H_2(X; \mathbb{R})$ . Then there exists a symplectic structure  $\omega'$  on  $X$ , deformation equivalent to  $\omega$ , such that all of these Lagrangian surfaces become symplectic. We can choose the symplectic structure  $\omega'$  such that the induced volume forms on the Lagrangians have any given sign. We can also assume that any symplectic surface disjoint from the Lagrangians stays symplectic.*

*Proof.* Let  $a_1, \dots, a_n$  be any real numbers. Since  $H^2(X; \mathbb{R})$  is the dual space of second real homology there exists a closed 2-form  $\eta$  on  $X$  such that

$$\int_{L_i} \eta = a_i, \quad i = 1, \dots, n.$$

Choose volume forms  $\omega_i$  on  $L_i$  for each  $i$  such that

$$\int_{L_i} \omega_i = \int_{L_i} \eta.$$

Let  $j_i$  denote the embedding of  $L_i$  into  $X$ . There exist 1-forms  $\alpha_i$  on  $L_i$  such that

$$\omega_i - j_i^* \eta = d\alpha_i.$$

Let  $\pi_i : \nu L_i \rightarrow L_i$  denote tubular neighbourhoods and choose cut-off functions  $\rho_i(r)$  with support on the tubular neighbourhoods which depend only on the radius  $r$  and are 1 on the zero section. Define 1-forms

$$\bar{\alpha}_i = \rho_i \pi_i^* \alpha_i$$

on the tubular neighbourhoods. Extend them by zero outside of the neighbourhood and set

$$\eta' = \eta + \sum_i d\bar{\alpha}_i.$$

We claim that  $j_i^* \eta' = \omega_i$ . This follows if we can show that

$$j_i^* d\bar{\alpha}_k = 0 \quad \text{for } k \neq i.$$

This is clear if  $L_k$  does not intersect  $L_i$  by making the tubular neighbourhood of  $L_k$  small enough so that it does not intersect  $L_i$ . Suppose that  $L_k$  and  $L_i$  intersect in a point  $p$ . We can assume that  $L_i$  intersects  $\nu L_k$  in a disk fibre of the tubular neighbourhood. We have

$$d\bar{\alpha}_k = \rho'_k dr \wedge \pi_k^* \alpha_k + \rho_k \pi_k^* d\alpha_k.$$

By assumption,  $\pi_{k*}$  is the zero map on  $T_q L_i$  for each point  $q$  on the disk fibre  $L_i \cap \nu L_k$ . Therefore  $d\bar{\alpha}_k$  is zero on any two vectors in  $T_q L_i$ . Hence  $j_i^* d\bar{\alpha}_k = 0$ .

Consider the closed 2-form

$$\omega' = \omega + t\eta'.$$

For small positive  $t$  the form  $\omega'$  is symplectic. Since the  $L_i$  are Lagrangian for  $\omega$  we have  $j_i^* \omega' = t\omega_i$ . Hence the  $L_i$  are now symplectic surfaces with (small) positive or negative volume, depending on the sign of  $a_i$ .  $\square$

**Definition 12.** Let  $\omega_0$  denote a symplectic form on  $M$  given by Proposition 9. We can assume that the symplectic form has the same sign on both  $R$  and  $S$ . Let  $T$  denote the symplectic torus of square 0 obtained by smoothing the intersection between  $R$  and  $S$ . The tori  $R$  and  $T$  intersect in a single positive transverse point.

The surfaces  $R$  and  $T$  together span a copy of  $H$  in the intersection form, which we denote by  $H_{RT}$ . In summary the intersection form of  $M$  is equal to

$$Q_M = H(n) \oplus H_{RT} \oplus aH \oplus b(-E_8)$$

with certain integers  $a, b \geq 1$ .

**Definition 13.** We say that a self-diffeomorphism of  $M$  satisfies  $(*)$  if it preserves the splitting  $H(n) \oplus H(n)^\perp$  and is the identity on the first summand.

We will frequently use the following proposition that was proved in [11].

**Proposition 14.** *Every integral class in  $H(n)^\perp$  can be mapped to any integral linear combination of  $R$  and  $T$  of the same square and divisibility by a self-diffeomorphism of the elliptic surface  $M$  that satisfies  $(*)$ . Taking a multiple we see that we can map in this way any rational class in  $H(n)^\perp$  to a rational linear combination of  $R$  and  $T$ .*

The following is clear:

**Lemma 15.** *Let  $f : M \rightarrow M$  be an orientation preserving diffeomorphism. If  $C$  and  $D$  are homology classes on  $M$  with  $f_*C = D$ , then  $(f^{-1})^*PD(C) = PD(D)$ .*

We will now cover a large part of the positive cone by symplectic forms in the following way: We have a symplectic form  $\omega_0$  so that the surfaces  $F, W, R$  and  $T$  are symplectic. The class of  $\omega_0$  can be written as

$$\omega_0 = PD(\alpha_0 F + \beta_0 W + \gamma_0 R + \delta_0 T) + Z_0,$$

where  $Z_0$  is a class in the real span of  $aH \oplus b(-E_8)$ . Using inflation with very large parameters and then dividing by a large number it follows that the class

$$\omega = PD(\alpha F + \beta W + \gamma R + \delta T)$$

plus some arbitrarily small rest is represented by a symplectic form for all positive coefficients  $\alpha, \beta, \gamma, \delta$ . The second method we use are the actions of self-diffeomorphisms on cohomology. In particular, we can map according to Proposition 14 any rational class in  $H^2(M; \mathbb{R})$  using a self-diffeomorphism to a rational linear combination of the Poincaré duals of  $F, W, R$  and  $T$ . This will suffice to prove Theorem 7 in Section 4, because in this situation all coefficients are positive. To prove the more general Theorem 4 in Section 5 we will introduce in Lemma 17 another diffeomorphism that allows in some situations to change a negative coefficient in the expansion of  $\omega$  into a positive one.

#### 4. PROOF OF THEOREM 7 ON POSITIVE CLASSES

We have the following lemma that proves one of the steps outlined above.

**Lemma 16.** *Let  $\omega$  be a class in  $\mathcal{P}^{PD(F)}$ . Then there exist a sequence of self-diffeomorphisms  $\phi_k$  of the elliptic surface  $M$  and classes  $\sigma_k$  of the form*

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T)$$

*with  $\beta > 0$  such that  $\phi_k^* \sigma_k$  converges to the class  $\omega$ . The diffeomorphisms  $\phi_k$  satisfy (\*). If  $\omega$  is a class in the subset  $\mathcal{P}^{PD(F)+}$  then we can assume that all coefficients of  $\sigma_k$  are positive.*

*Proof.* We decompose the class  $\omega$  as

$$\omega = PD(\alpha F + \beta W) + \omega',$$

where  $\omega' \in H(n)^\perp$  and  $\beta > 0$ . There exists a sequence  $\omega'_k$  of rational classes in  $H(n)^\perp$  converging to the class  $\omega'$ . Using the second part of Proposition 14 there exist self-diffeomorphisms  $\phi_k$  that satisfy (\*) and map

$$\phi_k^* PD(\gamma_k R + \delta_k T) = \omega'_k$$

for certain rational numbers  $\gamma_k, \delta_k$ . Setting

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T)$$

we get the first claim. If  $\omega$  is a class in  $\mathcal{P}^{PD(F)+}$  we can assume that all  $\omega_k'^2$  are positive. Hence we can assume that  $\gamma_k$  and  $\delta_k$  are positive.  $\square$

Recall that we have a symplectic form  $\omega_0$ . As above, the class of this form can be written as

$$\omega_0 = PD(\alpha_0 F + \beta_0 W + \gamma_0 R + \delta_0 T) + Z_0,$$

where  $Z_0$  is a class in the real span of  $aH \oplus b(-E_8)$ . We now prove Theorem 7.

*Proof.* Let  $\omega$  be a class in  $\mathcal{P}^{PD(F)+}$ . Choose a sequence  $\sigma_k$  as in Lemma 16. Then

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T)$$

where all coefficients are positive. Consider the symplectic form  $\omega_0$  with the symplectic surfaces  $F, W, R, T$ . We apply the inflation Lemma 8 to the form  $\omega_0$  which means that we can add to  $\omega_0$  any linear combination of the classes  $F, W, R, T$  with positive coefficients. This shows that the class

$$N_k \sigma_k + Z_0$$

is represented by a symplectic form for any sufficiently large positive number  $N_k$ . Hence also the classes

$$\eta_k = \sigma_k + \frac{1}{N_k} Z_0$$

are represented by symplectic forms. We know that  $\phi_k^* \sigma_k$  converges to  $\omega$ . We can choose the numbers  $N_k$  large enough so that  $\frac{1}{N_k} \phi_k^* Z_0$  converges to 0. Then  $\phi_k^* \eta_k$  converges to  $\omega$ , hence  $\omega \in \overline{\mathcal{C}}_M$ .  $\square$

## 5. PROOF OF THE MAIN THEOREM 4

We will use the following lemma which shows that certain automorphisms of the intersection form are realized by self-diffeomorphisms.

**Lemma 17.** *For an integer  $i$  let  $f_i$  denote the map which is the identity on all summands of the intersection form except on  $H(n) \oplus H_{RT}$ , where it is given by*

$$F \mapsto F$$

$$W \mapsto W + iT$$

$$R \mapsto R - iF$$

$$T \mapsto T.$$

*Then  $f_i$  is induced by a self-diffeomorphism of  $M$ .*

*Proof.* It is easy to check that  $f_i$  is an automorphism of the intersection form. The map  $f_i$  leaves  $F$  and hence  $c_1(M)$  invariant. Letting  $i$  be a real number and taking  $i \rightarrow 0$  we see that  $f_i$  has spinor norm one. This implies the claim by the work of Friedman-Morgan [7]; see also [16].  $\square$

We denote a diffeomorphism that induces  $f_i$  by the same symbol. The induced automorphism  $f_i^*$  on cohomology maps

$$\omega = PD((\alpha - i\gamma)F + \beta W + \gamma R + (\delta + i\beta)T)$$

to

$$f_i^* \omega = PD(\alpha F + \beta W + \gamma R + \delta T).$$



Note that the class  $\omega$  can be positive even if  $f_i^*\omega$  is not positive. The main difficulty in the case of Theorem 4 is that we have to approximate classes which are no longer positive by symplectic forms. However, the automorphism  $f_i^*$  allows us in some cases to map a positive class to such a non-positive class. The positive class can then be reached by inflation. Hence we have to show that under our assumptions we can always find an integer  $i$  such that  $f_i^*$  maps a positive class to our given class.

Suppose for example that we want the class  $\omega$  as above to be positive. We can assume that  $\beta, \gamma > 0$ . Then  $\omega$  is positive if and only if  $\alpha - i\gamma > 0$  and  $\delta + i\beta > 0$ . This is possible only if

$$\alpha\beta + \gamma\delta > 0,$$

which is equivalent to  $\omega^2 > 0$  if  $M$  is spin and  $\omega^2 > \beta^2$  if  $M$  is non-spin. Note that  $\beta = \omega \cdot PD(F)$ . This is the reason why we have to restrict to the subset  $\mathcal{P}^{PD(F)>}$  in the non-spin case.

We now begin with the proof of Theorem 4. Fix a cohomology class  $\omega$  in  $H^2(M; \mathbb{R})$ . If  $M$  is the elliptic surface  $E(2m)$  assume that  $\omega$  is in the subset  $\mathcal{P}^{PD(F)}$  and if  $M$  is the surface  $E(2m-1)$  assume that  $\omega$  is in the subset  $\mathcal{P}^{PD(F)>}$ . We want to approximate  $\omega$  by symplectic classes. Write

$$\omega = PD(\alpha F + \beta W) + \omega'$$

where  $\omega'$  is an element of the real span of  $H(n)^\perp$ . The following inequality for the coefficients of the class  $\omega$  is a consequence of our assumptions.

**Lemma 18.** *We have*

$$\alpha > -\frac{\omega'^2}{2\beta}.$$

*Proof.* In both cases  $\beta > 0$  and

$$\begin{aligned} 0 < \omega^2 &= 2\alpha\beta + \beta^2 W^2 + \omega'^2 \\ &= 2\alpha\beta + \epsilon(n)\beta^2 + \omega'^2 \end{aligned}$$

where  $\epsilon(n) = 0$  if  $n$  is even and  $\epsilon(n) = 1$  if  $n$  is odd. If  $n = 2m$  is even we get

$$2\alpha\beta > -\omega'^2$$

hence

$$\alpha > -\frac{\omega'^2}{2\beta}.$$

If  $n = 2m - 1$  is odd we get by the assumption that  $\omega$  is in  $\mathcal{P}^{PD(F)>}$

$$\beta^2 < \omega^2 = 2\alpha\beta + \beta^2 + \omega'^2.$$

This again implies the claim.  $\square$

We now prove a slightly technical lemma. The estimate in (b) will be used in Lemma 20 to show that we can find integers  $i_k$  such that the automorphisms  $f_{i_k}^*$  map a sequence of positive classes to another sequence which can then be mapped by diffeomorphisms to a sequence converging to our given class  $\omega$ .

**Lemma 19.** *There exists a sequence  $\omega'_k$  of rational classes in  $H(n)^\perp$  converging to  $\omega'$  with the following properties:*

- (a)  $\omega_k'^2 > \omega'^2$  for all indices  $k$ .
- (b) Write  $\omega'_k = \frac{1}{A_k}\tau_k$  where  $A_k$  is a positive rational number and  $\tau_k$  is an indivisible integral class in  $H(n)^\perp$ . Then there exist integers  $i_k$  with

$$\alpha > \frac{i_k}{A_k} > -\frac{\omega'^2}{2\beta}.$$

*Proof.* Let  $\omega_k''$  be any rational sequence in  $H(n)^\perp$  converging to  $\omega'$ . We can assume that

$$\omega_k''^2 > \omega'^2$$

because every neighbourhood of  $\omega'$  contains rational elements with this property. Write

$$\omega_k'' = \frac{1}{B_k}\mu_k$$

where  $B_k$  is a positive rational number and  $\mu_k$  is integral and indivisible. For each  $k$  we can find an integral basis  $e_1, e_2, \dots, e_r$  of the lattice  $H(n)^\perp$  such that  $e_1 = \mu_k$ . The basis depends on  $k$  but we do not write the index. Let  $C_k$  be an arbitrary sequence of positive integers converging to infinity. Consider the rational number  $A_k = C_k B_k$  and the integral class  $\tau_k = C_k \mu_k + e_2$ . Then  $\tau_k$  is indivisible. Define

$$\omega'_k = \frac{1}{A_k}\tau_k = \frac{1}{B_k}\left(\mu_k + \frac{1}{C_k}e_2\right).$$

If we choose the integers  $C_k$  large enough the sequence  $\omega'_k$  converges to  $\omega'$  (note that  $e_2$  depends on  $k$ ). Moreover, we can assume that  $\omega_k'^2 > \omega'^2$ . If  $C_k$  and hence  $A_k$  is large enough we can find an integer  $i_k$  such that

$$\alpha > \frac{i_k}{A_k} > -\frac{\omega'^2}{2\beta}.$$

□

Let  $\omega'_k = \frac{1}{A_k}\tau_k$  be the sequence from Lemma 19. Since  $\tau_k$  is an integral indivisible class in  $H(n)^\perp$  we can find by Proposition 14 a self-diffeomorphism  $\phi_k$  of the elliptic surface  $M$  satisfying (\*) such that

$$\tau_k = \phi_k^* PD(R + \delta_k T)$$

for certain integers  $\delta_k$ . We get

$$(1) \quad \omega'_k = \phi_k^* PD\left(\frac{1}{A_k}R + \frac{\delta_k}{A_k}T\right).$$

This implies that the sequence

$$\phi_k^* PD\left(\alpha F + \beta W + \frac{1}{A_k}R + \frac{\delta_k}{A_k}T\right)$$

converges to our given class  $\omega$ . Consider the automorphism  $f_i^*$  from Lemma 17 and apply  $(f_i^{-1})^*$  to the sequence

$$PD \left( \alpha F + \beta W + \frac{1}{A_k} R + \frac{\delta_k}{A_k} T \right)$$

where  $i = i_k$  for the integer  $i_k$  from Lemma 19. This implies that there exist self-diffeomorphisms  $\psi_k = f_{i_k} \circ \phi_k$  such that  $\psi_k^* \sigma_k$  converges to  $\omega$ , where

$$\sigma_k = PD \left( \left( \alpha - \frac{i_k}{A_k} \right) F + \beta W + \frac{1}{A_k} R + \left( \frac{\delta_k}{A_k} + i_k \beta \right) T \right)$$

**Lemma 20.** *The numbers  $\alpha - \frac{i_k}{A_k}$  and  $\frac{\delta_k}{A_k} + i_k \beta$  are positive.*

*Proof.* The first claim is clear by construction in Lemma 19. Note that by formula (1) above

$$\omega_k'^2 = \frac{2}{A_k^2} \delta_k$$

and by construction

$$\frac{i_k}{A_k} > -\frac{\omega_k'^2}{2\beta} > -\frac{\omega_k'^2}{2\beta}.$$

Hence

$$\frac{\delta_k}{A_k} = \frac{1}{2} \omega_k'^2 A_k.$$

and

$$i_k \beta > -\frac{1}{2} \omega_k'^2 A_k$$

This implies the second claim.  $\square$

Note that all coefficients of  $\sigma_k$  are positive. We now argue as in the proof of Theorem 7: There exist classes

$$\eta_k = \sigma_k + \frac{1}{N_k} Z_0$$

represented by symplectic forms such that  $\psi_k^* \eta_k$  converges to  $\omega$ . Hence  $\omega \in \overline{\mathcal{C}}_M$ . This proves Theorem 4.

## REFERENCES

1. D. Auroux, V. Muñoz, F. Presas, *Lagrangian submanifolds and Lefschetz pencils*, J. Symplectic Geom. **3**, 171–219 (2005).
2. P. Biran, *A stability property of symplectic packing*, Invent. math. **136**, 123–155 (1999).
3. J. Bowden, *Symplectic 4-manifolds with fixed point free circle actions*, preprint arXiv:1206.0458.
4. J. G. Dorfmeister, T.-J. Li, *The relative symplectic cone and  $T^2$ -fibrations*, J. Symplectic Geom. **8**, 1–35 (2010).
5. S. Friedl, S. Vidussi, *Twisted Alexander polynomials detect fibered 3-manifolds*, Ann. of Math. **173**, 1587–1643 (2011).
6. S. Friedl, S. Vidussi, *A vanishing theorem for twisted Alexander polynomials with applications to symplectic 4-manifolds*, preprint arXiv:1205.2434.
7. R. Friedman, J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 27, Springer-Verlag, Berlin, 1994.

8. H. Geiges, *Symplectic structures on  $T^2$ -bundles over  $T^2$* , Duke Math. J. **67**, 539–555 (1992).
9. R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142**, 527–595 (1995).
10. R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. Providence, Rhode Island. American Mathematical Society 1999.
11. M. J. D. Hamilton, *The minimal genus problem for elliptic surfaces*, preprint arXiv:1206.1260v1.
12. F. Lalonde, D. McDuff, *The classification of ruled symplectic 4-manifolds*, Math. Res. Lett. **3**, no. 6, 769–778 (1996).
13. T.-J. Li, *The space of symplectic structures on closed 4-manifolds*, AMS/IP Studies in Advanced Mathematics 42, 259–273 (2008).
14. T.-J. Li, A.-K. Liu, *Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with  $b^+ = 1$* , J. Differential Geom. **58**, 331–370 (2001).
15. T.-J. Li, M. Usher, *Symplectic forms and surfaces of negative square*, J. Symplectic Geom. **4**, 71–91 (2006).
16. M. Lönne, *On the diffeomorphism groups of elliptic surfaces*, Math. Ann. **310**, 103–117 (1998).
17. D. McDuff, *Notes on ruled symplectic 4-manifolds*, Trans. Amer. Math. Soc. **345**, 623–639 (1994).
18. C. H. Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2**, no. 2, 221–238 (1995).

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